Homework I
Generative Modeling by Transport: Mathematical Foundations

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Problem 1. A Regression Principle Hidden Inside a Geometry Problem Let X be uniformly distributed on the unit circle $S^1 \subset \mathbb{R}^2$. Let $Y \in \mathbb{R}^2$ be an integrable random vector satisfying

$$\mathbb{E}[Y \mid X = x] \cdot x = 0 \qquad \text{for almost every } x \in S^1.$$

For a measurable vector field $u : S^1 \to \mathbb{R}^2$, define

$$J[u] = \mathbb{E}\left[\frac{1}{2}|u(X)|^2 - Y \cdot u(X)\right].$$

(a) Prove that J is minimized by $u_\star(x) = \mathbb{E}[Y \mid X = x]$.

(b) Prove that the minimizer is tangent to the circle almost everywhere.

(c) Now suppose one only minimizes over tangent vector fields $u(x) \cdot x = 0$. Prove that the same minimizer is obtained.

(d) Explain why this is the correct abstract mechanism behind the quadratic objectives for velocities and denoisers in stochastic interpolants.

Problem 2. Gaussian Smoothing Without Saying "Diffusion" Let Y be an arbitrary $\mathbb{R}^d$-valued random variable, and let $Z \sim \mathcal{N}(0, I)$ be independent of Y. For a positive definite matrix $A \in \mathbb{R}^{d \times d}$, define

$$X = Y + AZ.$$

(a) Prove that X has a strictly positive C^∞ density.

(b) Express its characteristic function in terms of the characteristic function of Y.

(c) Prove that for every multi-index α, $\partial^\alpha \rho_X$ is bounded.

(d) Show that if $A_n \to 0$, then ρ_{Y+A_nZ} may lose smoothness in the limit.

(e) Explain why this is the analytic role of $\gamma(t)z$ in stochastic interpolants.

Problem 3. A Conservation Law From a Moving Random Cloud Let U be a random variable on a probability space, and let $X_t = F(t, U) \in \mathbb{R}^d$, where $F \in C^1([0,1] \times \Omega; \mathbb{R}^d)$. Suppose X_t has a strictly positive smooth density $\rho(t, x)$ for every t. Define

$$b(t,x) = \mathbb{E}[\partial_t F(t,U) \mid X_t = x].$$

(a) Prove that ρ satisfies

$$\partial_t \rho + \nabla \cdot (b\rho) = 0$$

in the weak sense.

(b) Give an example where b is not equal to $\partial_t F(t,U)$ as a pointwise function of x, but is a conditional average.

(c) Suppose $F(t,U) = a(t)U$, where $U \sim \rho_0$, $a(t) > 0$. Compute $b(t,x)$ explicitly.

(d) Verify directly that the density of $a(t)U$ solves the continuity equation with your b.

Problem 4. A Likelihood Formula on a Compact World Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the circle. Let $b \in C^1([0,1] \times \mathbb{T})$, and let $X_{s,t}(x)$ solve

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{s,t}(x) = b(t, X_{s,t}(x)), \qquad X_{s,s}(x) = x.$$

Let ρ solve

$$\partial_t \rho + \partial_x (b\rho) = 0.$$

(a) Prove that along characteristics,

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \rho(t, X_{s,t}(x)) = -\partial_x b(t, X_{s,t}(x)).$$

(b) Derive the circle analogue of the continuous change-of-variables formula.

(c) Suppose $b(t,x) = a(t) \sin x$. Express the likelihood of x_1 in terms of the backward characteristic.

(d) Explain why compactness removes boundary terms in the weak formulation.

Problem 5. The Invisible Gauge Freedom of Velocity Fields Let $\rho(t,x) > 0$ be a smooth probability density on $[0,1] \times \mathbb{R}^d$ with sufficient decay at infinity. Suppose b satisfies

$$\partial_t \rho + \nabla \cdot (b\rho) = 0.$$

(a) Characterize all smooth vector fields $\tilde{b}$ that generate the same density path $\rho(t)$ through the same continuity equation.

(b) Show that the difference $\tilde{b} - b$ is of the form J/ρ, where $\nabla \cdot J = 0$.

(c) In $d = 1$, prove that under decay at infinity the velocity field is unique.

(d) In $d = 2$, prove that any compactly supported smooth stream function $\psi(t,x)$ generates another admissible velocity field

$$\tilde{b} = b + \frac{\nabla^\perp \psi}{\rho}, \qquad \nabla^\perp \psi = (\partial_{x_2}\psi, -\partial_{x_1}\psi).$$

(e) Interpret this as a "gauge freedom" and explain why stochastic interpolants choose a particular velocity by conditional expectation.