Homework II
Generative Modeling by Transport: Mathematical Foundations

Ziseok Lee

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Problem 1. Anisotropic Stochastic Interpolants Let

$$x_t = I(t, x_0, x_1) + A(t)z,$$

where $z \sim \mathcal{N}(0, I_d)$, $A(t) \in \mathbb{R}^{d\times d}$ is invertible for $0 < t < 1$, and $A(0) = A(1) = 0$. Assume all regularity and integrability conditions needed to differentiate under the expectation.

(a) Prove that x_t has a smooth positive density for $0 < t < 1$.

(b) Prove that

$$b(t, x) = \mathbb{E}[\partial_t I(t, x_0, x_1) + \dot{A}(t)z \mid x_t = x]$$

satisfies

$$\partial_t \rho + \nabla \cdot (b\rho) = 0.$$

(c) Derive the anisotropic score identity:

$$\nabla \log \rho(t, x) = -A(t)^{-\top}\mathbb{E}[z \mid x_t = x].$$

(d) Write the corresponding denoiser objective for $\eta_z(t, x) = \mathbb{E}[z \mid x_t = x]$.

Problem 2. A Non-Gaussian Latent Variable Let z have smooth positive density $r(z) = e^{-V(z)}/Z$ on $\mathbb{R}^d$, with $V \in C^1$, and let

$$x_t = I(t, x_0, x_1) + \gamma(t)z, \qquad \gamma(t) > 0.$$

Let $\rho(t)$ be the density of x_t.

(a) Prove the generalized score identity

$$\nabla \log \rho(t, x) = -\gamma(t)^{-1}\mathbb{E}[\nabla V(z) \mid x_t = x].$$

(b) Recover the Gaussian formula as the special case $V(z) = |z|^2/2$.

(c) Construct a quadratic objective whose minimizer is $\mathbb{E}[\nabla V(z) \mid x_t = x]$.

(d) Explain why Gaussian latents are especially convenient for stochastic interpolants.

Problem 3. Endpoint Velocity With a Singular Latent Schedule Let

$$x_t = (1-t)x_0 + tx_1 + \gamma(t)z, \qquad \gamma(t) = \sqrt{a\,t(1-t)},$$

where $a > 0$, $z \sim \mathcal{N}(0, I)$, and $(x_0, x_1) \sim \nu$. Assume all regularity conditions.

(a) Show that

$$b(t,x) = v(t,x) - \gamma(t)\dot{\gamma}(t)s(t,x),$$

where $v(t,x) = \mathbb{E}[x_1 - x_0 \mid x_t = x]$.

(b) Compute $\gamma(t)\dot{\gamma}(t)$.

(c) Derive the endpoint limits $b(0,x)$ and $b(1,x)$.

(d) Specialize to the independent coupling $\nu = \rho_0 \otimes \rho_1$, and simplify the conditional expectations at the endpoints.

Problem 4. State-Dependent Diffusion Coefficients Let $\rho(t,x) > 0$ satisfy

$$\partial_t \rho + \nabla \cdot (b\rho) = 0.$$

Let $A(t,x)$ be a smooth symmetric positive semidefinite matrix field. Find a drift b^A such that

$$\partial_t \rho + \nabla \cdot (b^A \rho) = \frac{1}{2} \nabla \cdot \nabla \cdot (A\rho),$$

where

$$\nabla \cdot \nabla \cdot (A\rho) = \sum_{i,j} \partial_{ij}(A_{ij}\rho).$$

(a) Derive b^A in terms of b, A, ρ.

(b) Show that if $A = 2\varepsilon I$, your expression reduces to $b + \varepsilon \nabla \log \rho$.

(c) Explain why state-dependent diffusion requires an extra divergence correction.

Problem 5. Gaussianity Forced by Gaussian Bridges Let X_0, X_1, Z be independent real random variables, $Z \sim \mathcal{N}(0,1)$. Suppose that for every $t \in (0,1)$,

$$X_t = (1-t)X_0 + tX_1 + \sqrt{t(1-t)}\, Z$$

is Gaussian.

(a) Prove that $(1-t)X_0 + tX_1$ is Gaussian for every $t \in (0,1)$.

(b) Prove that X_0 and X_1 are Gaussian.

(c) Explain why this problem shows that Gaussian intermediate marginals are highly rigid.