Homework IV
Generative Modeling by Transport: Mathematical Foundations

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Problem 1. A Time-Dependent Diffusive Interpolant Let B_t be a Brownian bridge on $[0,1]$, and define

$$x_t^d = I(t, x_0, x_1) + \sqrt{2a(t)}B_t,$$

where $a(t) > 0$ is smooth. Assume B_t satisfies

$$\mathrm{d}B_t = -\frac{B_t}{1-t}\,\mathrm{d}t + \mathrm{d}W_t.$$

(a) Derive the SDE for x_t^d.

(b) Show that the marginal law agrees with the ordinary stochastic interpolant

$$x_t = I(t, x_0, x_1) + \sqrt{2a(t)t(1-t)}\,z.$$

(c) Derive a Fokker–Planck equation of the form

$$\partial_t\rho + \nabla\cdot(u\rho) = a(t)\Delta\rho.$$

(d) Express u as a conditional expectation.

Problem 2. Sampling From a Point Without Deterministic Transport Fix $x_* \in \mathbb{R}^d$. Let

$$x_t = x_* + \sqrt{2at(1-t)}z$$

for $t \in [0, \delta]$, and suppose the path is smoothly continued after δ to end at $x_1 \sim \rho_1$.

(a) Compute the score of x_t for $t \in (0, \delta]$.

(b) Compute the finite drift $u(t, x)$ in the Fokker–Planck equation

$$\partial_t \rho + \nabla \cdot (u\rho) = a\Delta\rho.$$

(c) Show that $u(t, x)$ has a finite limit as $t \downarrow 0$ at $x = x_*$.

(d) Prove that no smooth deterministic ODE can transport the point mass δ_{x_*} to a non-degenerate density in finite time.

Problem 3. One-Sided Interpolants With an Elliptical Gaussian Base Let $z \sim \mathcal{N}(0, I)$, let $C = \Sigma\Sigma^\top$ be positive definite, and define

$$x_t = \alpha(t)\Sigma z + J(t, x_1), \qquad x_1 \sim \rho_1.$$

(a) Derive the score identity in terms of $\eta_z(t, x) = \mathbb{E}[z \mid x_t = x]$.

(b) Derive the velocity field.

(c) Give the denoiser objective.

(d) Show how to recover the isotropic formula when $\Sigma = I$.

Problem 4. A Schrödinger Bridge Euler–Lagrange Puzzle Consider

$$\min_{\rho,u} \int_0^1 \int_{\mathbb{R}^d} \frac{1}{2}|u(t,x)|^2 \rho(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

subject to

$$\partial_t \rho + \nabla \cdot (u\rho) = \varepsilon \Delta \rho, \qquad \rho(0) = \rho_0, \quad \rho(1) = \rho_1.$$

(a) Introduce a Lagrange multiplier λ and derive the stationarity condition in u.

(b) Derive the viscous Hamilton–Jacobi equation for λ.

(c) Explain why $u = \nabla \lambda$.

(d) What formal limit is obtained as $\varepsilon \downarrow 0$?

Problem 5. A Mirror Bridge Cannot Stand Still Let $X \sim \rho$ on $\mathbb{R}^d$, where $\rho > 0$ is smooth and integrable. Let

$$x_t = X + \gamma(t)Z, \qquad Z \sim \mathcal{N}(0, I),$$

where $\gamma(t) > 0$ on an open interval $J \subset (0,1)$ and $\dot{\gamma}(t) \neq 0$ on J. Suppose the probability-flow velocity of this mirror path vanishes:

$$b(t,x) = 0 \qquad \forall (t,x) \in J \times \mathbb{R}^d.$$

(a) Show that this implies $\mathbb{E}[Z \mid X + \gamma(t)Z = x] = 0$.

(b) Use the score identity to prove that $\nabla \log \rho_t(x) = 0$ for all x.

(c) Conclude that no probability density on $\mathbb{R}^d$ can satisfy the assumption.

(d) Explain why the corresponding statement becomes possible on a compact torus with uniform density.