Homework V
Generative Modeling by Transport: Mathematical Foundations

Ziseok Lee

May 13, 2026

Problem 1. A Hidden Linear System of Conditional Means Let

$$x_t = \alpha(t)x_0 + \beta(t)x_1 + \gamma(t)z$$

with $z \sim \mathcal{N}(0, I)$. Define

$$\eta_0 = \mathbb{E}[x_0 \mid x_t = x], \qquad \eta_1 = \mathbb{E}[x_1 \mid x_t = x], \qquad \eta_z = \mathbb{E}[z \mid x_t = x].$$

(a) Prove the identity

$$\alpha\eta_0 + \beta\eta_1 + \gamma\eta_z = x.$$

(b) Derive the velocity field b in terms of η_0, η_1, η_z.

(c) Derive the score in terms of η_z.

(d) Suppose $\gamma \neq 0$. Express b only in terms of η_0, η_1, x.

Problem 2. Designing a Covariance-Preserving Bridge Let x_0, x_1, z be independent, centered random vectors in $\mathbb{R}^d$, each with covariance I. Let

$$x_t = \alpha(t)x_0 + \beta(t)x_1 + \gamma(t)z.$$

(a) Prove that x_t has identity covariance if and only if

$$\alpha(t)^2 + \beta(t)^2 + \gamma(t)^2 = 1.$$

(b) Among all smooth functions with $\alpha(0) = 1, \alpha(1) = 0, \beta(0) = 0, \beta(1) = 1, \gamma \geq 0$, propose a trigonometric design satisfying the covariance constraint.

(c) Compute $\gamma\dot{\gamma}$ for your design.

(d) Explain how $\gamma\dot{\gamma}$ controls endpoint score terms.

Problem 3. Gaussian Mixture Posterior Means Let x_0, x_1, z be independent. Suppose

$$x_0 \mid i \sim \mathcal{N}(m_i^0, C_i^0), \qquad x_1 \mid j \sim \mathcal{N}(m_j^1, C_j^1),$$

with mixture weights p_i^0, p_j^1. Let

$$x_t = \alpha x_0 + \beta x_1 + \gamma z.$$

(a) Compute the conditional law of x_t given mixture labels (i, j).

(b) Compute the posterior weights $w_{ij}(t, x)$.

(c) Compute $\eta_0(t, x)$, $\eta_1(t, x)$, and $\eta_z(t, x)$.

(d) Verify the algebraic constraint

$$\alpha\eta_0 + \beta\eta_1 + \gamma\eta_z = x.$$

Problem 4. One Denoiser to Rule Them All Consider the one-sided interpolant

$$x_t = \alpha(t)z + \beta(t)x_1, \qquad z \sim \mathcal{N}(0, I).$$

Assume $\beta(t) \neq 0$ for $t > 0$.

(a) Derive the identity

$$\alpha\eta_z + \beta\eta_1 = x.$$

(b) Express η_1 in terms of η_z and x.

(c) Derive the ODE velocity b using only η_z.

(d) Apply your result to $\alpha(t) = \sqrt{1 - t^2}$, $\beta(t) = t$, and simplify.

Problem 5. A Gaussianity Rigidity Theorem for Spatially Linear Designs Let X, Y, Z be independent real random variables, $Z \sim \mathcal{N}(0,1)$. Suppose that for every triple (a,b,c) in a nonempty open subset of the sphere

$$a^2 + b^2 + c^2 = 1, \qquad c \neq 0,$$

the random variable

$$aX + bY + cZ$$

is standard Gaussian.

(a) Prove that $aX + bY$ is Gaussian for all such (a,b,c).

(b) Prove that X and Y are Gaussian.

(c) Prove that their variances must be 1.

(d) Explain the implication for covariance-preserving spatially linear interpolants.