

Homework VI
Generative Modeling by Transport: Mathematical Foundations

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Problem 1. Compressing an Infinite-Time OU Process Let

$$Z_\tau = e^{-\tau} X + \sqrt{1 - e^{-2\tau}} G, \quad G \sim \mathcal{N}(0, I).$$

Set $t = e^{-\tau}$.

- (a) Express Z_τ as a one-sided interpolant $x_t = \alpha(t)G + \beta(t)X$.
- (b) Derive the one-sided velocity in terms of $\eta_G = \mathbb{E}[G \mid x_t = x]$.
- (c) Show that the formula can be written without explicit singularity using $\eta_X = \mathbb{E}[X \mid x_t = x]$.
- (d) Explain why this differs from directly time-changing the OU SDE.

Problem 2. Denoising Iteration on a Nonuniform Grid Let $0 = t_0 < t_1 < \dots < t_N = 1$.
 For a one-sided interpolant

$$x_t = \alpha(t)z + \beta(t)x_1,$$

define the deterministic denoising update

$$X_{j+1} = \frac{\beta(t_{j+1})}{\beta(t_j)} X_j + \left(\alpha(t_{j+1}) - \alpha(t_j) \frac{\beta(t_{j+1})}{\beta(t_j)} \right) \eta_z(t_j, X_j).$$

- (a) Derive this update from $\mathbb{E}[x_{t_{j+1}} | x_{t_j}]$.
- (b) Assume the mesh size tends to zero. Prove that the limiting ODE is

$$\dot{X}_t = \frac{\dot{\beta}}{\beta} X_t + \left(\dot{\alpha} - \frac{\alpha \dot{\beta}}{\beta} \right) \eta_z(t, X_t).$$

- (c) Explain why this is the same as the one-sided probability-flow ODE.

Problem 3. Rectification and Kinetic Action Let $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a smooth diffeomorphism, and define

$$M_t(x) = (1 - t)x + tT(x).$$

Assume M_t is a diffeomorphism for all t . The rectified path is

$$X_t = M_t(X_0).$$

- (a) Derive the rectified velocity b_t .
- (b) Prove that the terminal map is still T .
- (c) Show that the kinetic action of this path is

$$\int_0^1 \mathbb{E}|\dot{X}_t|^2 dt = \mathbb{E}|T(X_0) - X_0|^2.$$

- (d) In one dimension, explain why the monotone rearrangement minimizes this action among all maps transporting ρ_0 to ρ_1 .

Problem 4. Antithetic Cancellation Near Endpoints Let $F \in C^3(\mathbb{R}^d)$, $y \in \mathbb{R}^d$, and $z \sim \mathcal{N}(0, I)$. Define

$$A_\gamma(z) = \frac{F(y + \gamma z) - F(y - \gamma z)}{2\gamma}.$$

(a) Show that

$$A_\gamma(z) = \nabla F(y) \cdot z + O(\gamma^2 |z|^3).$$

(b) Suppose $F_z(x) = z \cdot s(x)$. Derive the leading term of

$$\frac{z \cdot s(y + \gamma z) - z \cdot s(y - \gamma z)}{2\gamma}.$$

(c) Explain why antithetic sampling stabilizes endpoint score losses.

Problem 5. Straight Lines Do Not Imply Optimal Transport Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation by angle $\theta \in (0, \pi)$, and let ρ_0 be the standard Gaussian. Since $T_{\#}\rho_0 = \rho_0$, T is a transport map from ρ_0 to itself. Consider the rectified straight path

$$X_t = (1 - t)X_0 + tT(X_0).$$

(a) Show that every sample path is a straight line.

(b) Compute the kinetic action

$$\mathbb{E}|T(X_0) - X_0|^2.$$

(c) Compare with the identity map.

(d) Conclude that straight sample paths alone do not characterize optimal transport.