Homework VII
Generative Modeling by Transport: Mathematical Foundations

Ziseok Lee

May 13, 2026

Problem 1. A Weighted Heat Flow That Becomes a Probability Path Let $\tilde{p}_t$ solve the unnormalized PDE

$$\partial_t \tilde{p}_t = \frac{1}{2}\Delta \tilde{p}_t + g_t \tilde{p}_t, \qquad \tilde{p}_0 = p_0.$$

Let

$$Z_t = \int \tilde{p}_t(x)\,\mathrm{d}x, \qquad p_t = \tilde{p}_t / Z_t.$$

(a) Derive the normalized PDE for p_t.

(b) Prove the Feynman–Kac formula

$$\int \varphi p_T = \frac{\mathbb{E}\left[e^{\int_0^T g_s(B_s)\,\mathrm{d}s}\varphi(B_T)\right]}{\mathbb{E}\left[e^{\int_0^T g_s(B_s)\,\mathrm{d}s}\right]},$$

where B_t starts from p_0.

(c) Explain why adding a purely time-dependent function $c(t)$ to g_t changes no normalized expectations.

Problem 2. A Three-Expert Geometric Average Let $q_t^{(1)}, q_t^{(2)}, q_t^{(3)}$ solve the same reverse diffusion marginal PDE, with scores $s_i = \nabla \log q_t^{(i)}$. Let

$$p_t^\star \propto \prod_{i=1}^{3} q_t^{(i)\,\gamma_i}, \qquad \gamma_1 + \gamma_2 + \gamma_3 = 1.$$

Simulate the heuristic SDE with score

$$s^\star = \sum_{i=1}^{3} \gamma_i s_i.$$

(a) Derive the FKC correction g_t in terms of the s_i.

(b) Show that

$$g_t = -\frac{\sigma_t^2}{2}\left(\sum_i \gamma_i |s_i|^2 - \left|\sum_i \gamma_i s_i\right|^2\right).$$

(c) Specialize to two experts with $\gamma_1 = 1 - \beta, \gamma_2 = \beta$.

(d) Explain why the correction vanishes when one exponent is 1 and the rest are 0.

Problem 3. Converting Transport and Diffusion Into Weights Let p_t be a positive density with score $s_t = \nabla \log p_t$.

(a) Find g^{flow} such that

$$-\nabla \cdot (v p_t) = g^{\text{flow}} p_t.$$

(b) Find g^{diff} such that

$$\frac{\sigma_t^2}{2} \Delta p_t = g^{\text{diff}} p_t.$$

(c) Use these formulas to derive a weighted particle system for a PDE with both transport and diffusion but no actual movement.

Problem 4. A Jump Process That Implements Reweighting Let p be a probability density and g an integrable function. Define

$$c(x) = g(x) - \mathbb{E}_p[g].$$

Let

$$c^+(x) = \max(c(x), 0), \qquad c^-(x) = \max(-c(x), 0).$$

(a) Prove

$$\int c^+ p = \int c^- p.$$

(b) Define a jump rate and transition kernel that realize

$$\partial_t p = cp.$$

(c) Prove the forward equation of your jump process equals cp.

(d) Interpret this process as a continuous-time version of SMC resampling.

Problem 5. Exact Normalizer for a Gaussian Product-Ratio Path Let

$$q_i(x) = \mathcal{N}(x; 0, \sigma_i^2 I_d), \qquad i = 1, \dots, m,$$

and let real exponents γ_i define

$$h(x) = \prod_{i=1}^{m} q_i(x)^{\gamma_i}.$$

(a) Determine necessary and sufficient conditions for $h \in L^1(\mathbb{R}^d)$.

(b) Compute the normalizer $Z = \int h(x)\,\mathrm{d}x$.

(c) Compute the score of $p = h/Z$.

(d) Suppose $m = 2$, $\gamma_1 = 1 - \beta, \gamma_2 = \beta$. Determine for which β the path is normalizable.