Homework VIII
Generative Modeling by Transport: Mathematical Foundations

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Problem 1. Path Existence for a Coordinate-Projected Ratio Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Suppose three experts act on coordinate sets

$$I_1 = \{1, 2\}, \qquad I_2 = \{2, 3\}, \qquad I_3 = \{1, 3\}.$$

At time t, each lifted expert has Gaussian-tail behavior

$$\tilde{q}_i(t, x) \asymp \exp\left(-\frac{\|\pi_i x\|^2}{2\alpha_i(t)^2} + O(\|x\|)\right).$$

Let

$$h_t(x) = \prod_{i=1}^{3} \tilde{q}_i(t, x)^{\gamma_i}.$$

(a) Derive the coordinate precision coefficients $C_1(t), C_2(t), C_3(t)$.

(b) Give a sufficient condition for $h_t \in L^1(\mathbb{R}^3)$.

(c) Give a sufficient condition for collapse.

(d) Explain why the condition depends on the hypergraph $\{I_i\}$.

Problem 2. Constructing an ACE Bump Let a scalar path-existence criterion be

$$C(t) = \frac{1}{(1-t)^2} - \frac{\omega}{\cos^2(\pi t/2)}, \qquad t \in [0, t_{\text{end}}],$$

where $t_{\text{end}} < 1$, $\omega > 0$. Suppose you may add a bump $b(t)$ to the positive exponent of the first term, producing

$$\tilde{C}(t) = \frac{1 + b(t)}{(1-t)^2} - \frac{\omega}{\cos^2(\pi t/2)}.$$

Let

$$b(t) = B\, t(1-t).$$

(a) Find a sufficient lower bound on B ensuring $\tilde{C}(t) \geq \delta > 0$ for all $t \in [\tau, t_{\text{end}}]$.

(b) Explain how to handle $t \in [0, \tau]$.

(c) Why does this bump preserve endpoints?

(d) What numerical trade-off appears when B is too large?

Problem 3. Deriving the ACE Weight for a Time-Varying Ratio Let

$$h_t(x) = \prod_{i=1}^{n} q_i(t,x)^{\gamma_i(t)}.$$

Assume each q_i satisfies a continuity equation

$$\partial_t q_i + \nabla \cdot (v_i q_i) = 0.$$

Let

$$s_i = \nabla \log q_i, \qquad s^\star = \sum_i \gamma_i s_i.$$

Choose a base velocity $v^\star$, and simulate a deterministic flow with velocity $v^\star$. We want a Feynman–Kac weight g_t so that the weighted density follows h_t/Z_t.

(a) Compute $\partial_t \log h_t$.

(b) Compare it with the log-density evolution generated by $v^\star$.

(c) Derive

$$g_t = \nabla \cdot v^\star + \sum_i \gamma_i D_i + \sum_i \dot{\gamma}_i \log q_i,$$

where

$$D_i = -\nabla \cdot v_i + (v^\star - v_i) \cdot s_i.$$

(d) Explain why the final term disappears for FKC.

Problem 4. Concentration From the Path Existence Criterion Assume a normalized density satisfies

$$p_t(x) \le A \exp\left(-\frac{1}{2}C\|x\|^2 + B\|x\|\right), \qquad C > 0.$$

(a) Prove that for $R \ge 4B/C$,

$$\mathbb{P}(\|X\| > R) \le A' \exp(-cCR^2)$$

for constants $A', c > 0$.

(b) Deduce that the $(1-\epsilon)$-quantile radius satisfies

$$R_\epsilon = O(C^{-1/2})$$

up to logarithmic factors.

(c) Interpret why $C(t) \downarrow 0$ destabilizes SMC weights.

Problem 5. A Matrix-Valued Path Existence Criterion Let experts have tail envelopes

$$\tilde{q}_i(t,x) \asymp \exp\left(-\frac{1}{2}x^\top P_i(t)x + O(\|x\|)\right),$$

where $P_i(t)$ are symmetric positive semidefinite matrices. Let

$$h_t(x) = \prod_i \tilde{q}_i(t,x)^{\gamma_i(t)}.$$

Define the effective precision matrix

$$P_{\text{eff}}(t) = \sum_i \gamma_i(t)P_i(t).$$

(a) Prove that if $P_{\text{eff}}(t)$ is positive definite, then $h_t \in L^1(\mathbb{R}^d)$.

(b) Prove that if $P_{\text{eff}}(t)$ has a negative eigenvalue, then $h_t \notin L^1(\mathbb{R}^d)$.

(c) Explain how the scalar coordinate criterion $C_k(t) > 0$ is recovered when all $P_i(t)$ are diagonal coordinate projections.

(d) Explain why the zero-eigenvalue case is delicate.